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J. Phys. A: Math. Gen. **15** (1982) 1701–1704. Printed in Great Britain

Linear fluctuations in a plasma

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Received 24 July 1981, in final form 2 November 1981

Abstract. The linear fluctuation spectrum is calculated from the Boltzmann equation with an arbitrary linear collision term.

In the last two decades the incoherent scatter technique has received much interest. Today there exists a variety of ways to calculate the density fluctuation spectrum of a plasma. Thus, Dougherty and Farley (1960, 1963) used the fluctuation dissipation theorem. Rosenbluth and Rostoker (1962) showed that, for a collisionless plasma, the spectrum can be calculated by means of superposition of shielded test particles. Hagfors and Brockelman (1971) presented a theory based on transition probability functions. The case of a non-Maxwellian electron velocity distribution function and electron–ion collisions was considered by Perkins and Salpeter (1965), who calculated the effect of suprathermal electrons on the electron plasma lines. In addition, we also mention the works of Grewal (1964a, b), Williamson (1968) and Theimer and Theimer (1973). In all of these theories weakly coupled systems are assumed, i.e. systems which are adequately described by only the particle self-motion and the mean fields. Collisions are included by means of a general linearised average collision operator C. The Boltzmann equation for the system thus reads

$$\left(g^{-1}+\frac{q}{m}\boldsymbol{E}(\boldsymbol{r},t)\cdot\frac{\partial}{\partial\boldsymbol{v}}\right)f(\boldsymbol{r},\boldsymbol{v},t)=0 \tag{1a}$$

where $\boldsymbol{E}(\boldsymbol{r},t)$ is the electrostatic field and the operator $g^{-1}=\partial/\partial t+\boldsymbol{v}\cdot\partial/\partial\boldsymbol{r}-C$ describes the particle self-motion. In Fourier–Laplace space thus

$$g^{-1}=\mathrm{i}(\omega-\boldsymbol{k}\cdot\boldsymbol{v})-C. \tag{1b}$$

Other notations are standard.

In several cases of practical interest (e.g. Perkins and Salpeter 1965, Ganguly *et al* 1979) we have to consider linearly stable plasmas, for which non-Maxwellian distribution functions and collisions both significantly affect the scattered spectrum. In a plasma with non-Maxwellian distribution functions, there are sometimes unstable modes, and our theory can of course not be used for such plasmas. However, stable collisional non-Maxwellian plasmas do occur (e.g. Perkins and Salpeter 1965, Ganguly *et al* 1979), and we shall restrict our attention to such plasmas.

Previous theories are not quite adequate for calculating the fluctuation spectrum of a collisional plasma if the velocity distribution functions are general. For instance, the general theory of Hagfors and Brockelman (1971) is somewhat restricted by the fact that their equation (39) has not yet been solved in the general case. The other

0305-4470/82/051701+04\$02.00 © 1982 The Institute of Physics

theories mentioned above, Dougherty and Farley (1960, 1963), Perkins and Salpeter (1965), Grewal (1964a, b), Williamson (1968), Theimer and Theimer (1973), only consider particular collision models or Maxwellian distribution functions. The purpose of the present paper is thus to present a simple calculation of the random spectrum of a plasma, governed by equation (1*a*), with arbitrary collision operator C and velocity distribution functions. The spectrum thus obtained will unify the results obtained in all the works quoted above, except for Theimer and Theimer (1973) in the case of a velocity-dependent collision frequency. After a number of minor corrections that paper will also agree with our result, however†.

We now derive the spectrum. This can be done by first neglecting the collective effects, i.e. setting the plasma parameter $\alpha = \omega_p/(kv_T)$ to zero. The spectrum for $\alpha \neq 0$ is then calculated in the same manner as in Grewal (1964a). Thus, for $\alpha = 0$, we choose some arbitrary initial state, where for any l, the lth particle in the system has the position $\boldsymbol{r}_l$ and the velocity $\boldsymbol{v}_l$ at the time $t = 0$. The space Fourier transform of the particle number density is then

$$n(\boldsymbol{k}, 0) = \sum_l \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{r}_l) \tag{2a}$$

at the time $t = 0$. The *average* evolution of all systems satisfying this initial condition is

$$n_{\mathrm{L}}(\boldsymbol{k}, \omega) = \sum_l \int \mathrm{d}\boldsymbol{v}\, g[\delta(\boldsymbol{v} - \boldsymbol{v}_l)] \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{r}_l) \tag{2b}$$

where the index L denotes the time Laplace transform and g is the inverse of the operator g^{-1}. The average of the quantity $n(-\boldsymbol{k}, 0)n_{\mathrm{L}}(\boldsymbol{k}, \omega)$, with the initial condition (2*a*) fixed, is thus

$$n(-\boldsymbol{k}, 0)n_{\mathrm{L}}(\boldsymbol{k}, \omega) = \sum_{l_1, l_2} \exp[-\mathrm{i}\boldsymbol{k} \cdot (\boldsymbol{r}_{l_1} - \boldsymbol{r}_{l_2})] \int \mathrm{d}\boldsymbol{v}\, g[\delta(\boldsymbol{v} - \boldsymbol{v}_{l_2})]. \tag{2c}$$

We finally average this expression over all initial states. This is done assuming that $\boldsymbol{r}_{l_1}$ and $\boldsymbol{r}_{l_2}$ are distributed uniformly in space and the velocities $\boldsymbol{v}_{l_2}$ are distributed according to the unperturbed velocity distribution function, f_0. A simple calculation then yields

$$2\,\mathrm{Re}\langle n(-\boldsymbol{k}, 0)n_{\mathrm{L}}(\boldsymbol{k}, \omega)\rangle = 2n_0\,\mathrm{Re} \int \mathrm{d}\boldsymbol{v}\, g(f_0) + (2\pi)^4 \delta(\boldsymbol{k})\delta(\omega) n_0^2 \tag{2d}$$

where n_0 is the unperturbed particle number density. The first term on the right-hand side corresponds to the terms with $l_1 = l_2$ in equation (2*c*) and the second one to those with $l_1 \neq l_2$. In the latter case we also use the fact that, when $\boldsymbol{k} = \boldsymbol{0}$, $\mathrm{Re}\, g(f_0) = \mathrm{Re}(-\mathrm{i}\omega^{-1} f_0) = \pi\delta(\omega) f_0$ (since ω must have a small imaginary part if the Laplace transform in equation (2*b*) is to converge). The dynamic form factor thus becomes

$$S^0(\boldsymbol{k}, \omega) = 2\,\mathrm{Re} \int \mathrm{d}\boldsymbol{v}\, g(f_0) \tag{2e}$$

where the superscript 0 on a quantity denotes that it has been calculated for $\alpha = 0$.

† Due to the lengthy algebra, the details cannot be presented here. These are stored under the British Library Supplementary Publications Scheme (PS reference number SUP 90057), and they can also be obtained from the author.

The restrictions on equation (2e) are obvious from the statistical assumptions used when deriving equation (2d). That is, apart from the *mean* interaction given by the operator C, the particles move independently of each other. These are exactly the assumptions of equation (1a), when the collective effects are neglected.

For $\alpha \neq 0$, the Fourier transform of the first-order density perturbation can be found, by solving the linearised equation (1a), in terms of the susceptibility

$$\chi(\boldsymbol{k}, \omega) = \mathrm{i}\frac{\omega_{\mathrm{p}}^{2}}{k^{2}} \int \mathrm{d}\boldsymbol{v}\, g\left(\boldsymbol{k} \cdot \frac{\partial f_0}{\partial \boldsymbol{v}}\right) \tag{3a}$$

and the corresponding perturbation for $\alpha = 0$, $n_1^0(\boldsymbol{k}, \omega)$ as

$$n_1(\boldsymbol{k}, \omega) = [1 + \chi(\boldsymbol{k}, \omega)]^{-1} n_1^0(\boldsymbol{k}, \omega). \tag{3b}$$

Using the relation

$$\langle n_1(\boldsymbol{k}, \omega) n_1^*(\boldsymbol{k}, \omega')\rangle = 2\pi\delta(\omega - \omega') N S(\boldsymbol{k}, \omega) \tag{3c}$$

where N is the number of particles in the system we derive

$$S(\boldsymbol{k}, \omega) = |1 + \chi(\boldsymbol{k}, \omega)|^{-2} S^0(\boldsymbol{k}, \omega). \tag{3d}$$

In equation (3d) it is assumed that the presence of collective effects does not affect the average $\langle n_1^0(\boldsymbol{k}, \omega) n_1^{0*}(\boldsymbol{k}, \omega')\rangle$. This is known as the dielectric superposition principle.

For Maxwellian velocity distribution functions, it is now a simple matter to verify that equations (2e), (3a) and (3d) satisfy the fluctuation dissipation theorem; and in particular (e.g. Sitenko 1967)

$$S^0(\boldsymbol{k}, \omega) = -2(\alpha^2\omega)^{-1} \operatorname{Im} \chi(\boldsymbol{k}, \omega). \tag{4}$$

The results of Hagfors and Brockelman (1971) are recovered by noticing that the transition probability function is simply the Green function to the operator g^{-1} (equation (39) in that paper being equivalent to the adjoint of equation (1a) with $\alpha = 0$). The spectrum of Perkins and Salpeter (1965) has been derived from equation (3d) elsewhere (Uddholm 1981).

An alternative relation to equation (4) has recently been suggested by Theimer and Behl (1980). This relation does not agree with the fluctuation dissipation theorem. On the other hand, we notice that, for a constant collision frequency, the spectrum calculated by Theimer and Theimer (1973) is the same as that derived *by means of* the fluctuation dissipation theorem, e.g. Dougherty and Farley (1963) or Sheffield (1975) for the simple BGK collision model. This discrepancy is due to the fact that Theimer and Behl (1980) calculated the response function by the method of Chambers (1952), who used the simple relaxation model. See also equations (13) and (27) in Hagfors and Brockelman (1971).

Only one species was treated above, but generalisation to plasmas with many species is straightforward. Thus

$$S(k, \omega) = \left|1 - \frac{\chi_{\mathrm{e}}}{\varepsilon}\right|^2 S_{\mathrm{e}}^0(\boldsymbol{k}, \omega) + \left|\frac{\chi_{\mathrm{e}}}{\varepsilon}\right|^2 \sum_{\sigma \neq \mathrm{e}} \frac{n_{\sigma 0}}{n_{\mathrm{e}0}} \left|\frac{q_\sigma}{q_{\mathrm{e}}}\right|^2 S_\sigma^0(\boldsymbol{k}, \omega) \tag{5}$$

where χ_σ, S_σ^0, $n_{\sigma 0}$ and q_σ are the susceptibility, the random density fluctuation spectrum, the unperturbed particle number density and the particle charge for species σ; $\sigma = \mathrm{e}$ for the electrons. The dielectric number $1 + \Sigma_\sigma \chi_\sigma$ is denoted by ε. Swartz and Farley

(1979) have recently presented a generalised version of the fluctuation dissipation theorem, valid for plasmas with the species having different temperatures and drift velocities. We notice that, for Maxwellian velocity distribution functions, equation (5) reduces to their result.

We have presented a derivation of the random spectrum in a plasma. Its validity is restricted to plasmas in which, instead of solving the many-body problem, collisions can be accounted for by an average collision operator *C*. In other words, the range of validity is the same as that of the Boltzmann equation (1*a*). Such plasmas are of great experimental interest. Collective effects were then included by means of the dielectric superposition principle. Generalisation to arbitrary external fields is, of course, quite straightforward.

The author thanks Professor L Stenflo for valuable discussions.

References

Chambers R G 1952 *Proc. Phys. Soc.* A **65** 458–9
Dougherty J P and Farley D T 1960 *Proc. R. Soc.* A **259** 79–99
—— 1963 *J. Geophys. Res.* **68** 5473–86
Ganguly S, Mathews J D and Tepley C A 1979 *Geophys. Res. Lett.* **6** 89–92
Grewal M S 1964a *Phys. Rev.* **134** A86–93
—— 1964b *Phys. Rev.* **136** A1181–6
Hagfors T and Brockelman R A 1971 *Phys. Fluids* **14** 1143–51
Perkins F W and Salpeter E E 1965 *Phys. Rev.* **139** A55–62
Rosenbluth M N and Rostoker N 1962 *Phys. Fluids* **5** 776–88
Sheffield J 1975 *Plasma Scattering of Electromagnetic Radiation* (London: Academic) pp 128–9
Sitenko A G 1967 *Electromagnetic Fluctuations in a Plasma* (New York: Academic) p 32
Swartz W E and Farley D T 1979 *J. Geophys. Res.* **84** 1930–2
Theimer O and Behl Y K 1980 *Phys. Fluids* **23** 292–5
Theimer O and Theimer R 1973 *Plasma Phys.* **15** 837–52
Uddholm P 1981 *Phys. Scr.* **22** 637–9
Williamson J H 1968 *J. Phys. A: Gen. Phys.* **1** 629–44